Week 1

Stable Envelopes: Compute examples

Approximately: (Cohomology)

A map

$$\mathrm{Stab}_\sigma : H_T(X^T) \longrightarrow H_T(X)$$
$$p \longmapsto \mathrm{Stab}_\sigma(p)$$

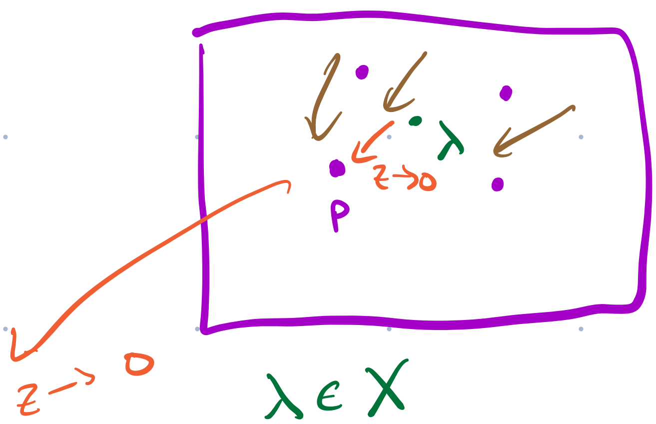
$\sigma \in \mathrm{Cochar}(T)$

Depends on choices $\mathbb{C}^\times \to T$, z coord, $\mapsto$ choice of direction of flow

$X = T^*\mathbb{P}^n, \; T^*Gr, \; Hilb^n(\mathbb{C}^2)$

3 Axioms:

X



$\lambda \in X$

$$\lim_{z\to 0} \sigma(z)(\lambda)$$

$$\mathrm{Attr}_\sigma(p) = \left\{ x \in X : \lim_{z\to 0} \sigma(z)(x) = p \right\}$$

$p =$ "1" in $H_T(p)$

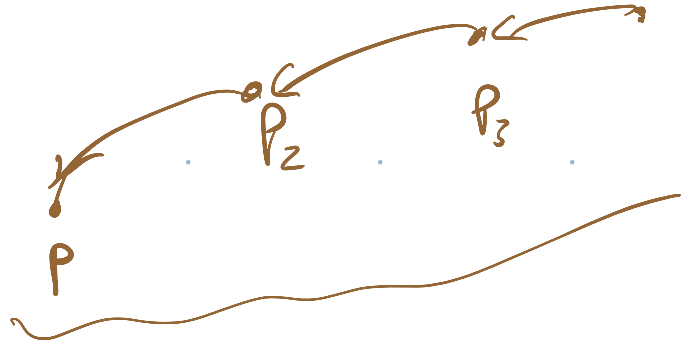
$X^T = \{p_1, \ldots, p_n\}$

$H_T(X^T) = \bigoplus_{i=1}^{n} H_T(p_i)$

$\mathbb{C}[T]$

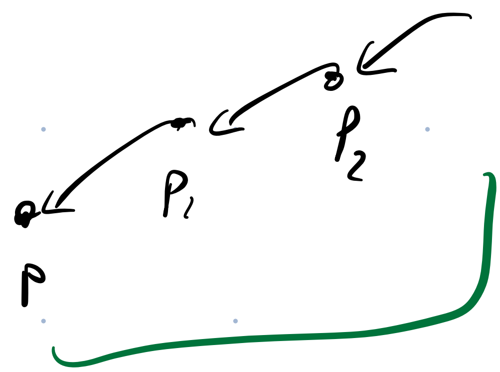
1

$Attr_\sigma^f(p)$ minimal set invariant wrt closure of $Attr(p)$



full attracting set

First axiom says (1) $Stab_\sigma(p)$ is supported at $Attr_\sigma^f(p)$



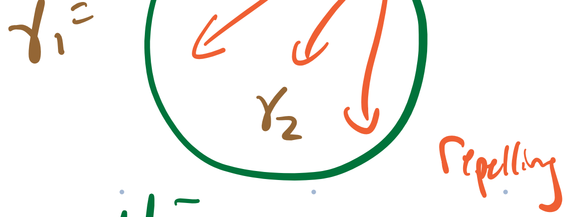
Only nontrivial restrictions on full attracting set

$$(2) \quad \underbrace{Stab_\sigma(p)|_p}_{\text{Equiv Coh Class}} = \pm e(N_p^-) \in H_T(p) = \mathbb{C}[T]$$

Euler class

"repelling" bundle

Euler class



N_p^- - vector space w/ torus action scaling

$$e(N_p^-) = \prod_{\gamma \in \mathrm{Char}_T(N_p^-)} \gamma$$

(3) $\deg_T \big(\underbrace{\mathrm{Stab}(p)|_{p'}}_{\text{polynomial}} \big) < \frac{1}{2} \dim X$

$$\deg \big(\mathrm{Stab}(p)|_p \big) = \frac{1}{2} \dim X$$

"4" $\mathrm{Stab}_\sigma(p) \in H_T(X)$

Okounkov-Maulik paper / Instanton R-matrix

Compute for $T^*\mathbb{P}^n$

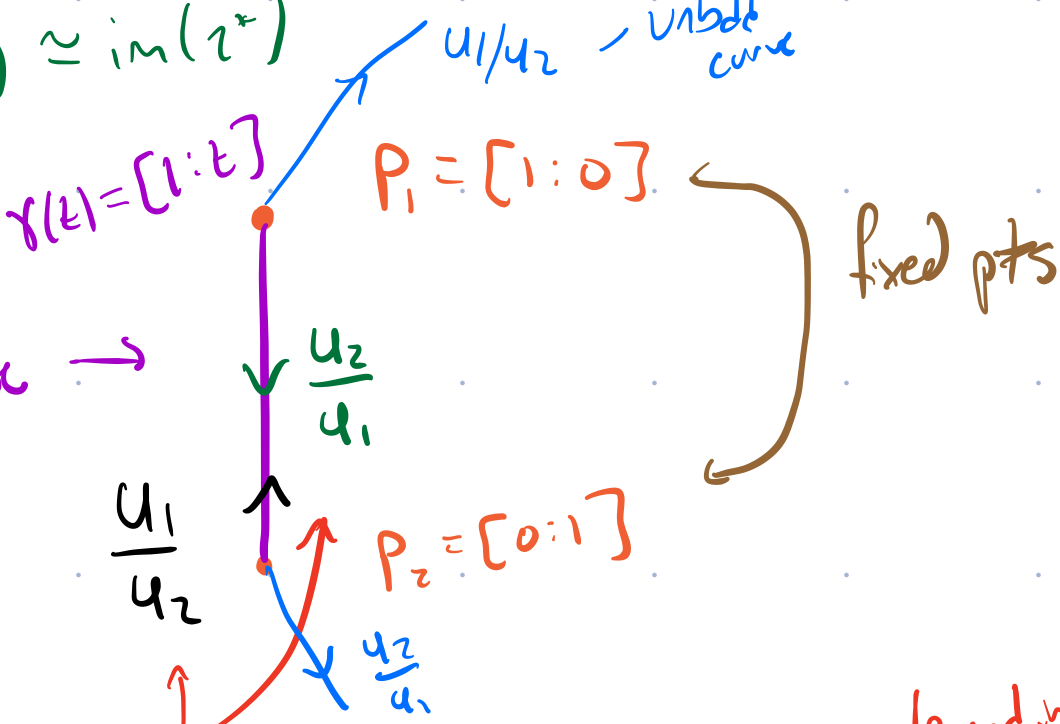
$$\underbrace{\mathrm{Stab}_\sigma(p_i)\big|_{p_j} = A_{ij}}_{\text{matrix of polys}}$$

9/11 Alex Computes

$X = T^*\mathbb{P}^1$, $T = (\mathbb{C}^\times)^2$

$$H_T^*(X) \xhookrightarrow{\iota^*} H_T^*(X^T) \simeq \bigoplus_{p \in X^T} \mathbb{Q}[u_1, u_2]$$

$$H_T^*(X) \simeq \mathrm{im}(\iota^*)$$



spheres look ccw & cw depending on what you're looking

Really $(P_1, 0) \in T^*\mathbb{P}^1$

$$\mathrm{im}(\iota^*) = \{(f_1, f_2) \in \mathbb{Q}[u_1, u_2]^2 \mid u_1 - u_2 \mid f_1 - f_2\}$$

Choose cocharacter

$$\sigma : \mathbb{C}^\times \longrightarrow T$$

$$\sigma(z) = (z, z^2)$$ this gives flows

Ex)

$$\lim_{z \to 0} \sigma(z) \cdot [1:1] = \lim_{z \to 0} (z, z^2) \cdot [1:1]$$

$$= \lim_{z \to 0} [1 : z] = [z : z^2]$$

copy of $\mathbb{C}$
affine coordinates

$$= [1:0] = p_1$$

$$\gamma \subseteq \mathrm{Attr}_\sigma(p_1) \qquad p_1 > p_2$$

$$p_2 \in \overline{\gamma} \subseteq \mathrm{Attr}_\sigma^f(p_1)$$

Axioms of Stable Envelope

$$\mathrm{Stab}_\sigma : H_T^*(X^T) \longrightarrow H_T^*(X)$$

$\cdot\, |_{p_1} \to \mathbb{Q}[u_1, u_2]$

$\cdot\, |_{p_2} \to \mathbb{Q}[u_1, u_2]$

$$H_T^*(X^T) \cong \mathbb{Q}[u_1,u_2] \oplus \mathbb{Q}[u_1,u_2]$$

$$[p_1] = (1,0) \qquad (0,1) = [p_2]$$

1) $\mathrm{Supp}(\mathrm{Stab}_\sigma(p)) \subseteq \mathrm{Attr}_\sigma^f(p)$ $\quad \left(\mathrm{Stab}_\sigma(p)|_{p'} = 0 \text{ where } p' > p \right)$

2) $\mathrm{Stab}_\sigma(p)|_p = \pm\, e(N_-) = \pm \prod\limits_{\substack{\text{weights} \\ \text{in } N_-}} w$

3) $\deg\left(\mathrm{Stab}_\sigma(p)|_{p'}\right) < \frac{1}{2}\dim X$

$p' < p$

e.g. $\bullet_{p_1} \xrightarrow{u_1 - u_2}$

$$e(N_-) = u_1 - u_2$$

$\mathrm{Stab}_\sigma(p_1)\big|_{p_1} = u_1 - u_2$

$\mathrm{Stab}_\sigma(p_2)\big|_{p_2} = u_1 - u_2$

$\mathrm{Stab}_\sigma(p_2)\big|_{p_1} = 0$

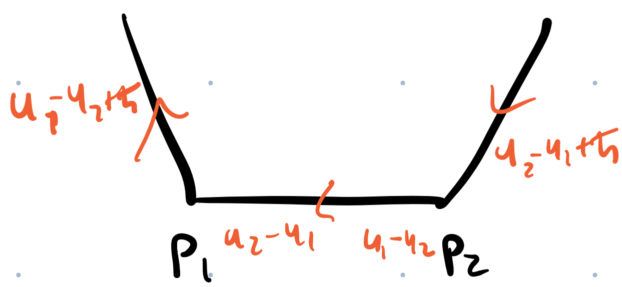
$\deg\left(\mathrm{Stab}_\sigma(p_1)\big|_{p_2}\right) < \frac{1}{2}\dim X = 1$

$\mathrm{Stab}_\sigma(p_1)\big|_{p_2} = n$

$u_1 - u_2 \,\big|\, \mathrm{Stab}(p_1)\big|_{p_1} - \mathrm{Stab}(p_1)\big|_{p_2}$

$u_1 - u_2 \,\big|\, (u_1 - u_2) - n$

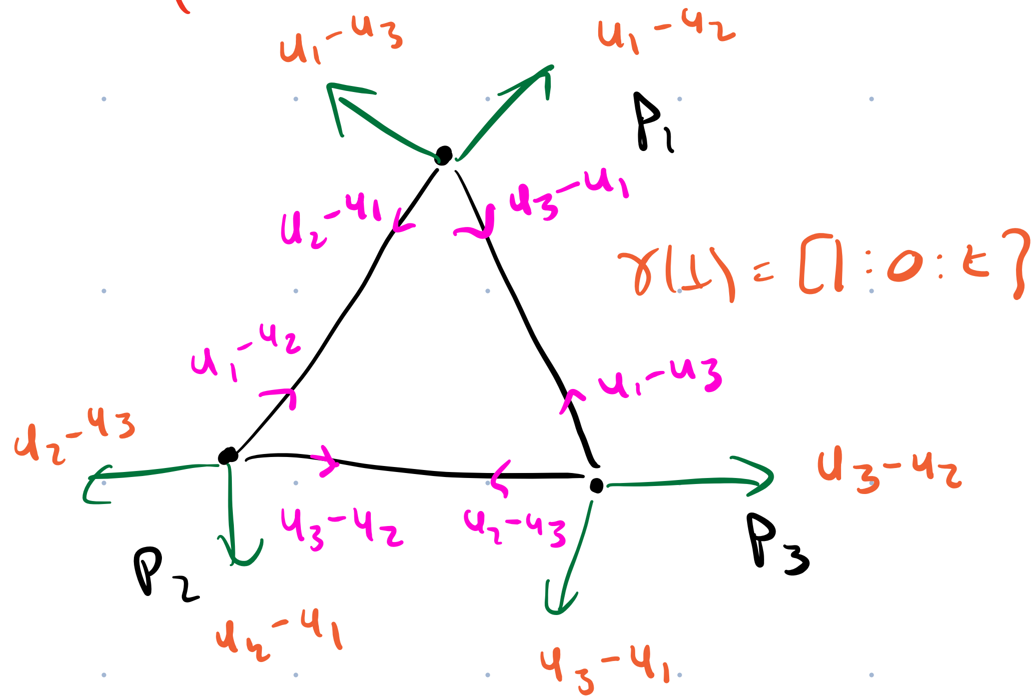
$\Rightarrow n = 0$



$$\Rightarrow \begin{pmatrix} u_1 - u_2 + \hbar & \hbar \\ 0 & u_1 - u_2 \end{pmatrix}$$

Ex $X = T^*\mathbb{P}^2$

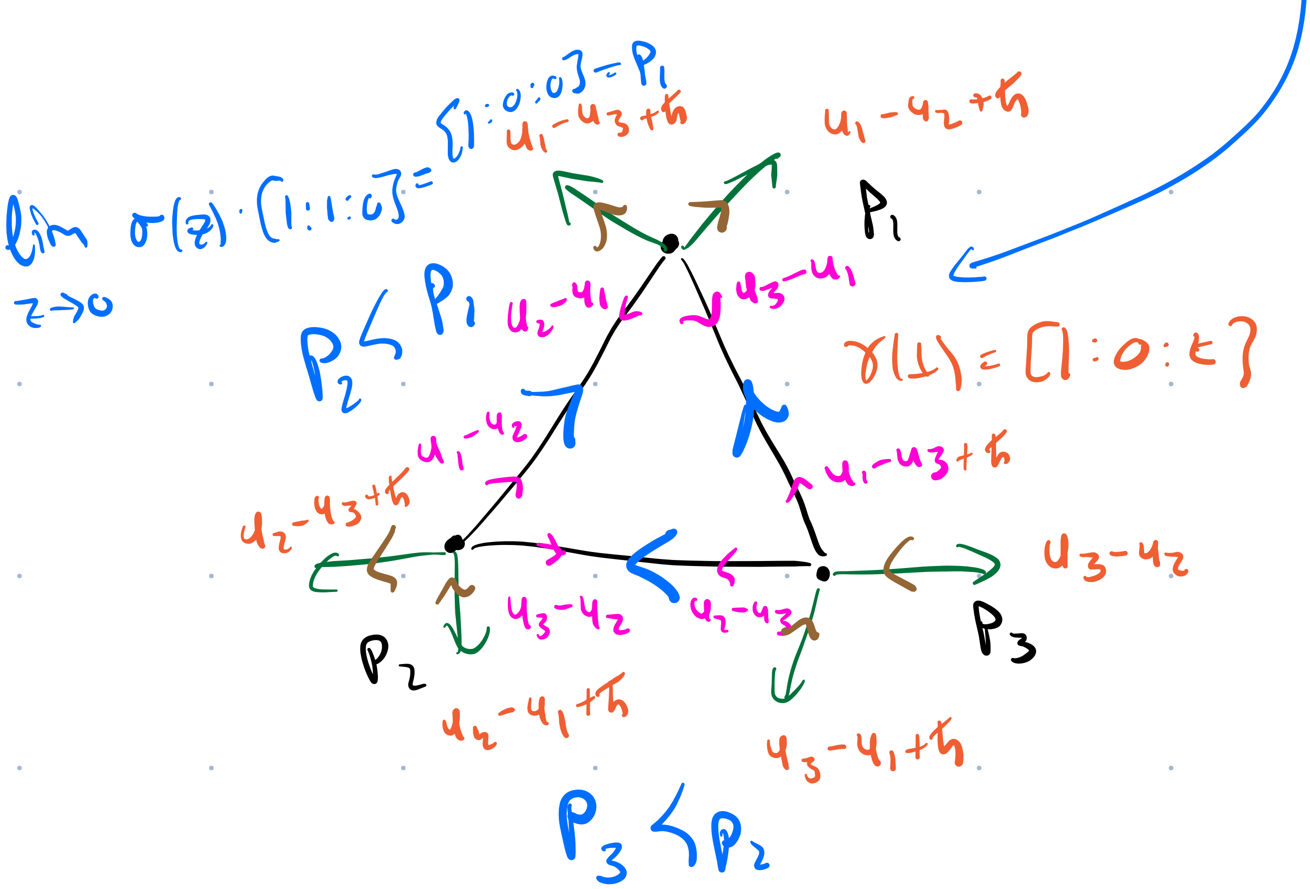
$T = (\mathbb{C}^\times)^3$



$\sigma : \mathbb{C}^\times \to T \qquad \sigma(z) = (z, z^2, z^3)$

$$\lim_{z\to 0} \sigma(z)\cdot[1:0:1] = P_1$$

$$\Rightarrow \quad \underline{P_3 \prec P_1}$$



$$\left[\mathrm{Stab}_\sigma(p_i)\big|_{p_j}\right]_{i,j=1}^{3} = \begin{pmatrix} (u_1-u_3+\hbar)(u_1-u_2+\hbar) & \hbar(u_1-u_3+\hbar) & \hbar(u_1-u_2+\hbar) \\ 0 & (u_1-u_2)(u_2-u_3+\hbar) & \hbar(u_1-u_2) \\ 0 & 0 & (u_1-u_3)(u_2-u_3) \end{pmatrix}$$

Row i = $\mathrm{Stab}_\sigma(p_i)$

$$u_2 - u_3 \mid \mathrm{Stab}(p_2)|_{p_2} - \mathrm{Stab}(p_2)|_{p_3}$$

$$\Rightarrow \quad u_2 - u_3 \mid \mathrm{Stab}(p_2)|_{p_3}$$

Gluing condition

$$\mathcal{F}|_{p_1} = \mathcal{F}|_{p_2}$$

Extra action of $\mathbb{C}^\times_\hbar$, Scales symplectic form

Scale fibe

$\deg_A \mathrm{Stab}(p_i)|_{p_j} < \frac{1}{2} \dim X$

↳

$T = A \times \mathbb{C}^\times_\hbar$

↳ Scales symplectic form

R - Matrix

$$R = \left(Stab_-\right)^{-1} Stab_+$$

reverse arrows

lower Δ

upper Δ

Open Problem:

$$\int_X Stab(p) = \sum_{q \in X^T} \frac{Stab(p)|_q}{e(T_q X)} \Bigg|_{u_1 = \cdots = u_n = 0}$$

$$\in \mathbb{Z}$$

T^*Flag

What is Compute for T^*Partial flag

Ex)

(first row of matrix)

$$\int_{T^*\mathbb{P}^1} \mathrm{Stab}(p_1) = \frac{u_1-u_2+\hbar}{(u_1-u_2+\hbar)(u_2-u_1)} + \frac{\hbar}{(u_1-u_2)(u_2-u_1+\hbar)}$$

$$= \left.\frac{-u_2+u_1-\hbar+\hbar}{(u_1-u_2)(u_2-u_1+\hbar)}\right|_{u_1=u_2=0}$$

$$= \frac{1}{\hbar}$$

$$= \hbar^{-1}\cdot \underbrace{1}_{\text{integer}}$$

$T^*\mathbb{P}^n$

$$\begin{array}{ccc} & 1\ 1 & \\ 1 & 2 & 1 \end{array}$$

T^*Gr unknown

$$H^*(T^*\mathbb{P}^n) \simeq H^*(\mathbb{P}^n) = \frac{\mathbb{Q}[c, u_1, \dots, u_n, \hbar]}{\prod (c-u_i)}$$

Cohomology clirs.

that restricts
to every pt, so easy guess
$c|_{p_i} = u_i$

"Weight function"

$$w_i(c, u_1, \dots, u_n, t_1)$$

$$w_i(c, \dots, t_1)|_{p_i} = w_i(c = u_j\ u_1 \dots u_n\ t_1)$$

$$= \mathrm{Stab}(p_i)|_{p_j}$$

$c|_{p_i} = u_i$

$T^*\mathbb{P}^1$

$$w_2 = -(c - u_1)$$

$$w_1 = c - u_2 + t_1$$

Questions to ask:

- Where do "unbounded curves" come from?

- Why is there the given directions in Unbounded curve : to preserve ω

$\omega = \sum dp_i \wedge dq_i$
$= \sum d(\frac{1}{\lambda})p_i \wedge d(\lambda p_i)$

- In $T^*\mathbb{P}^1$ what is $Attr_{\mathbb{C}}(0) / Attr_{\mathbb{C}}(\infty)$?

- How do I think about $\mathbb{C}[\mathrm{Lie}\, T]$?

- What does "degree" mean in the context of cohomology?

- What does the polarization do?

9/18 Reese

$$\mathbb{P}^2 \ni [x:y:z] \circlearrowleft (\mathbb{C}^\times)^3 \quad (\lambda_1, \lambda_2, \lambda_3) \Bigg\} \quad [\lambda_1 x : \lambda_2 y : \lambda_3 y]$$

$\exists$ induced action on $T^*\mathbb{P}^2$, in local coord there is formula

$$\underset{A}{(\mathbb{C}^\times)^3} \circlearrowright T^*\mathbb{P}^2 \quad \text{preserving symplectic form}$$

$$T = (\mathbb{C}^\times)^3 \times \mathbb{C}^\times_\hbar \circlearrowright T^*\mathbb{P}^2$$

Locally, T^*X has coordinates

$$(\underbrace{q_1, \ldots, q_n}_{\text{base space}}, \underbrace{p_1, \ldots, p_n}_{\text{differentials of coord}}) \quad \text{where} \quad p_i = dq_i$$

different d

In this nbhd $\omega = \sum dp_i \wedge dq_i$

$$\text{"} A \cdot \omega = \sum d(\ ) \wedge d(\lambda_i q_i)$$

$$= \sum d(\tfrac{1}{\lambda_i} q_i) \wedge \lambda_i (dq_i)$$

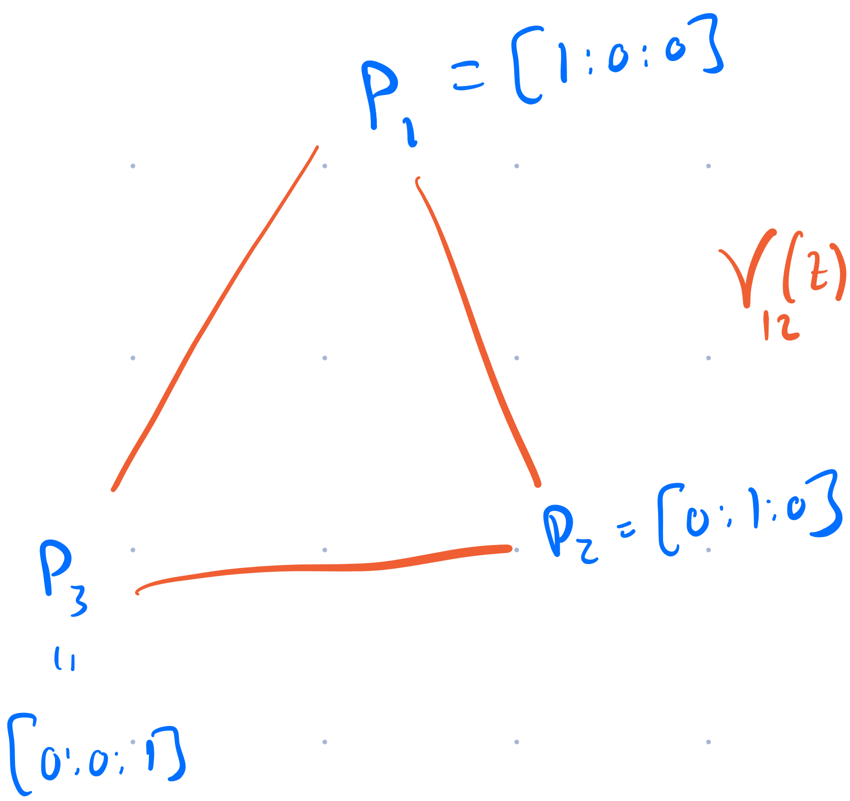
$$= \omega$$

$$H_T^\bullet(T^* \mathbb{P}^2) = \left\{ (f_1, f_2, f_3) \in \mathbb{C}[u_1, u_2, u_3, \hbar]^3 \;\middle|\; \begin{array}{l} u_1 - u_2 \mid f_1 - f_2 \\ u_2 - u_3 \mid f_2 - f_3 \\ u_1 - u_3 \mid f_1 - f_3 \end{array} \right\}$$

Moment Graph:

$\exists$ 3 Fixed pts of T-action

— - fixed curve



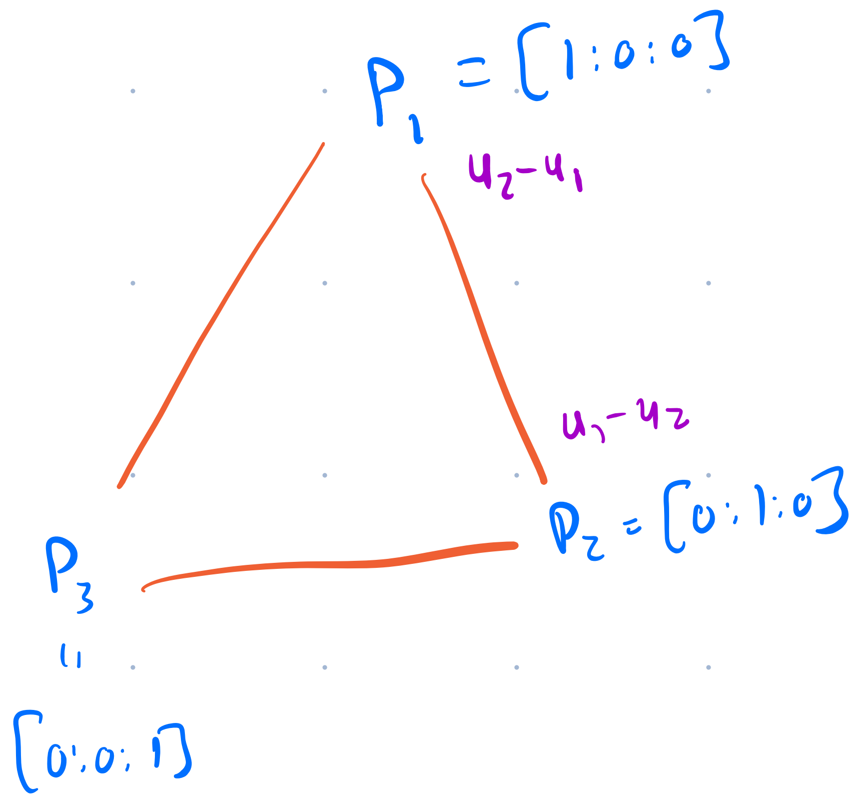
$\gamma_{12}(t) = [1:t:0]$

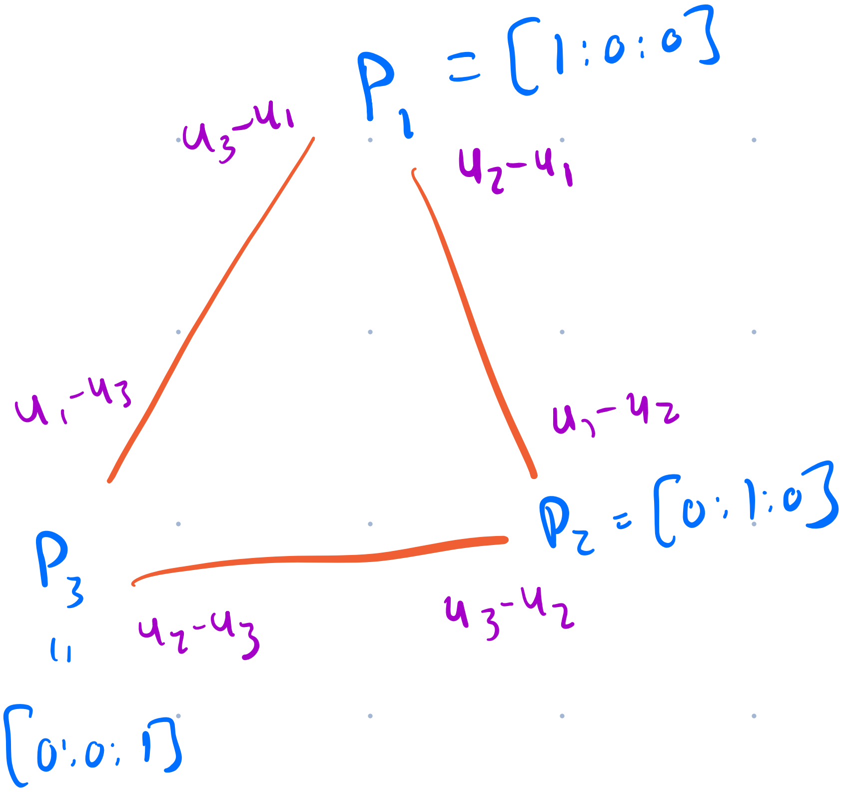
Near P_1 in the direction of P_2, coord are

$$[z_1 : z_2 : 0]$$

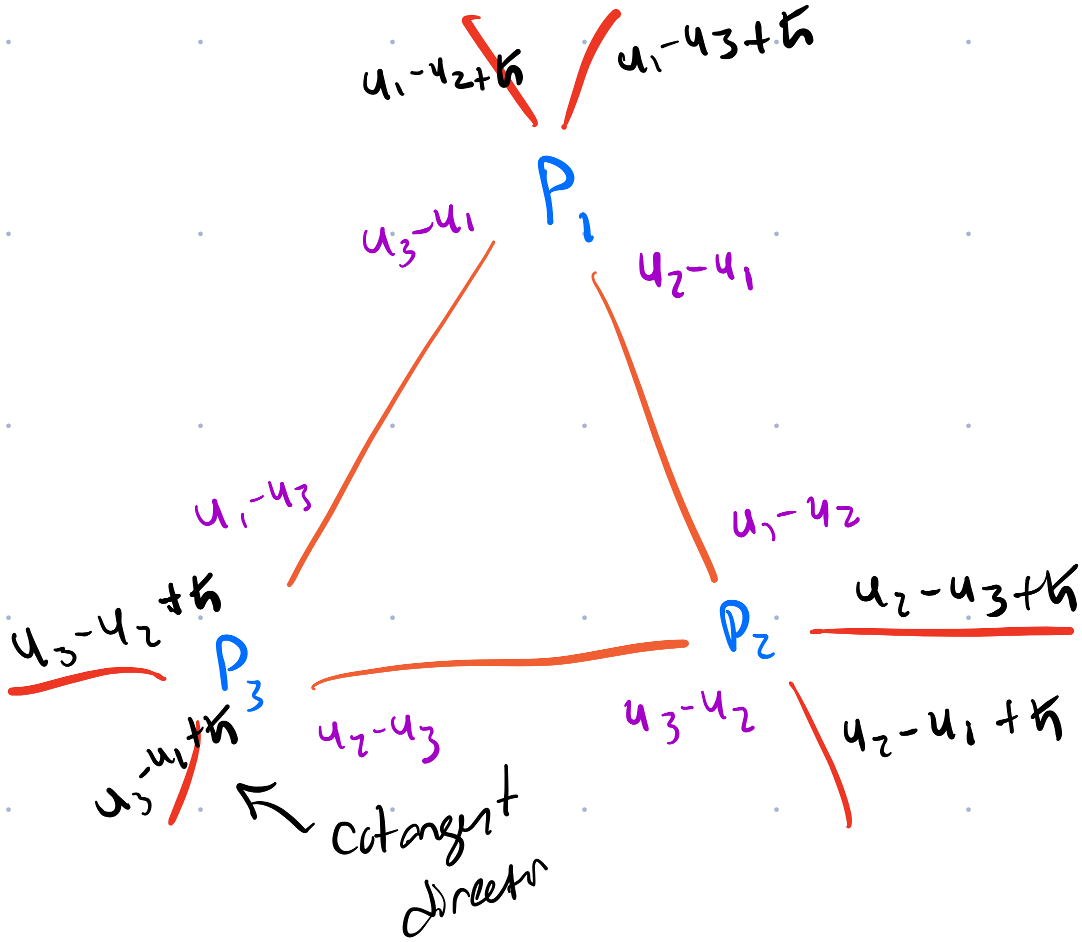
$$= \left[1 : \frac{z_2}{z_1} : 0\right]$$

$$T \cdot \left[1 : \frac{z_2}{z_1} : 0\right] = \left[1 : \frac{\lambda_2}{\lambda_1} \frac{z_2}{z_1} : 0\right]$$





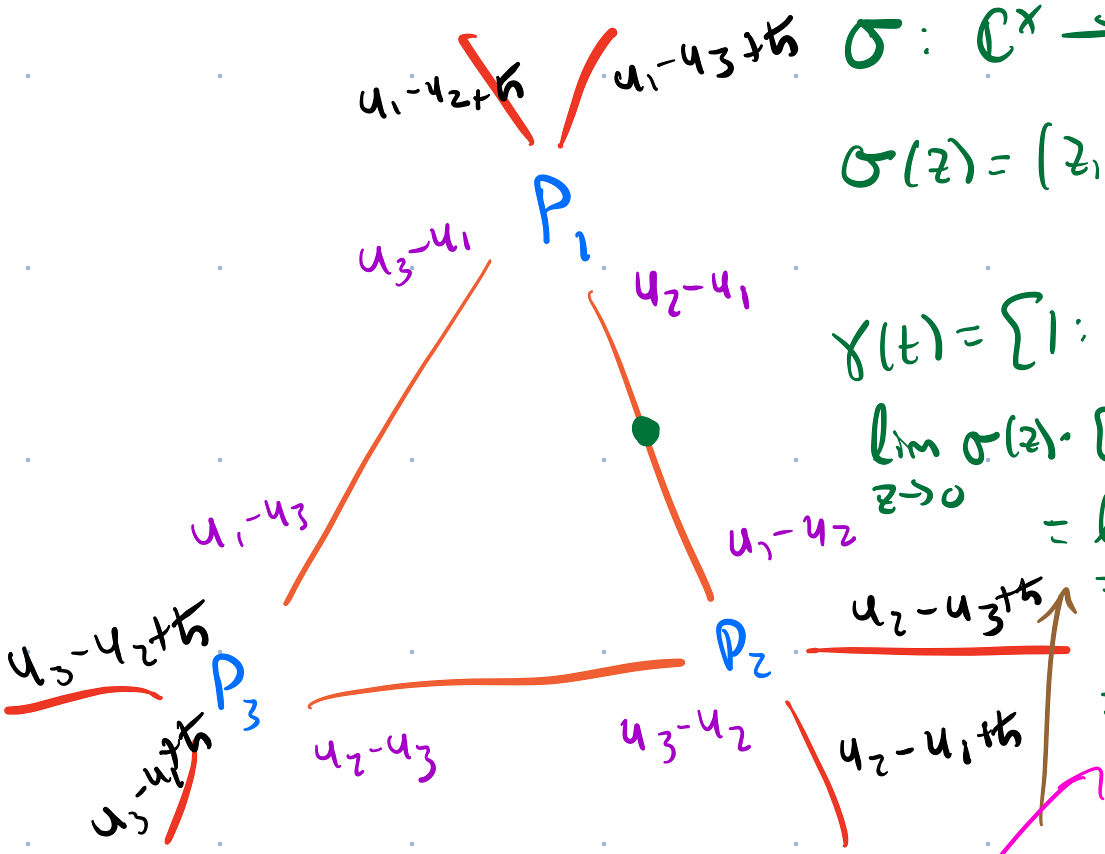
Co Tangent space is 4 dim near each pt



Cocharacter

$\sigma : \mathbb{C}^\times \to T$

$\sigma(z) = (z, z^2, z^3, 1)$



$\gamma(t) = [1 : t : 0]$

$\lim_{z\to 0} \sigma(z)\cdot[1:t:0]$

$= \lim_{z\to 0} [z : z^2 t : 0]$

$= \lim_{z\to 0} [1 : zt : 0]$

$= [1:0:0]$

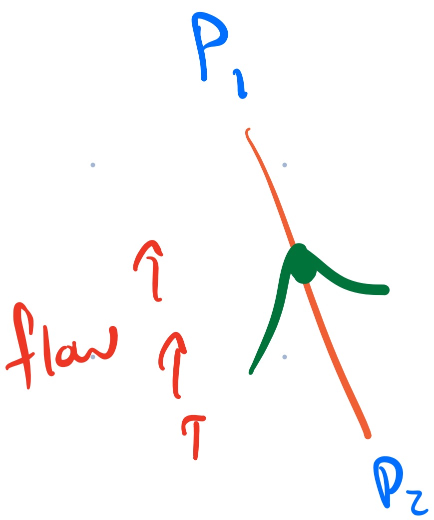
$= P_1$

Not affine space no limits!

affine chart

so limits works

So,

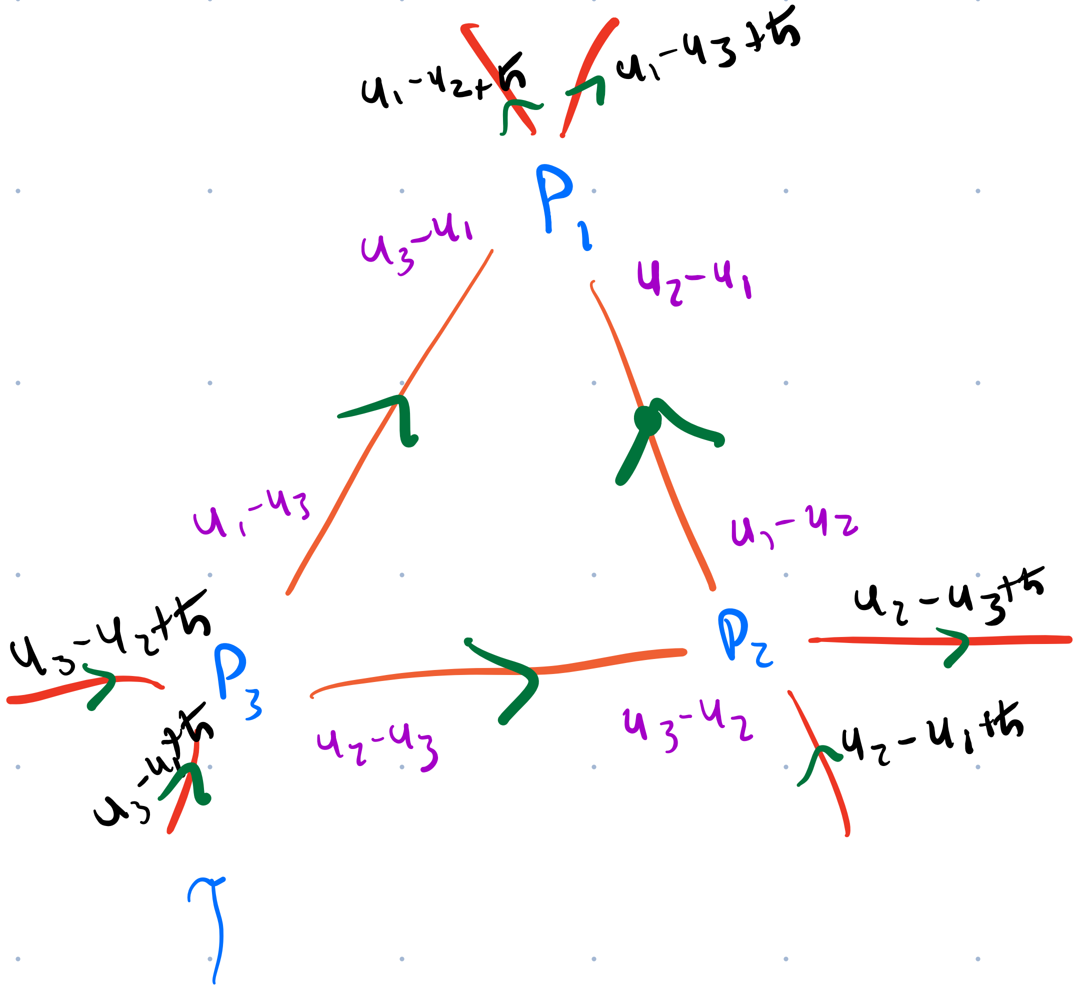


is the attracting direction

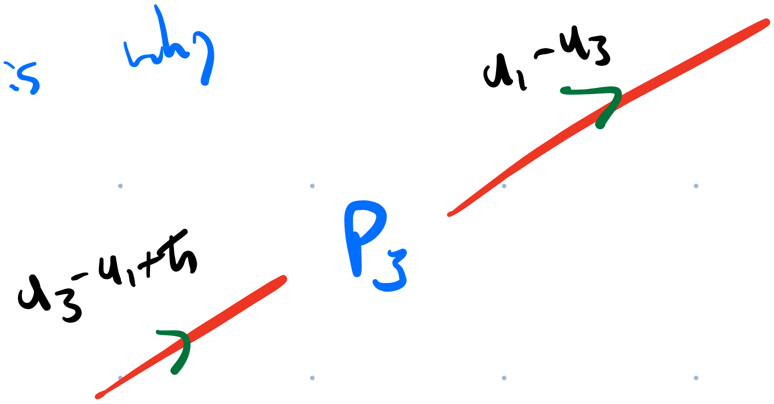
Since on the curve $\gamma(t)$

$\lim \sigma(z)\cdot\gamma(t) = P_1$

$z \to 0$



w/o $\hbar$
these curves
are the same as
A-rep (same weight)
so flow in same direction
this is why



Stable Envelopes:

X has fin many fixed pts / curves

$$\mathrm{Stab}: H_T^\bullet(X^T) \longrightarrow H_T^\bullet(X)$$

map of $H_T^\bullet(pt)$- modules

$$\Rightarrow H_T^\bullet(X^n) \simeq \bigoplus H_T^\bullet(pt)$$
$$= \bigoplus \mathbb{C}[u_1, \dots, u_d, \hbar]$$

p_i have classes $(0, \dots, 0, 1, 0, \dots, 0)$

$\mathrm{Stab}(p_i) \in H_T^\bullet(X)$

Also maps

$$H_T^\bullet(X) \xrightarrow{|_{p_i}} H_T^\bullet(pt)$$

$$\mathrm{Stab}(p_i)\big|_{p_j}$$

Axioms

1) $\mathrm{Supp}(\mathrm{Stab}(p_i)) \subset \mathrm{Attr}_\sigma^F(p_i)$

2) $\mathrm{Stab}(p_i)|_{p_i} = e(N_-)|_{p_i}$

repelling directions

3) $\mathrm{Stab}(p_i)|_{p_j} < \frac{1}{2} \dim X$

$p_j \subset \mathrm{Attr}_\sigma^F(p_i)$

4) $\mathrm{Stab}(p_1) = (f_1, f_2, f_3)$

$u_1 - u_2 \mid f_1 - f_2$

$\Longleftrightarrow \quad f_1|_{u_1=u_2} = f_2|_{u_1=u_2}$

$$\mathrm{Stab}(p_1)|_{p_1} = (u_1 - u_2 + \hbar)(u_1 - u_3 + \hbar)$$

$$\mathrm{Stab}(p_2)|_{p_2} = (u_1 - u_2)(u_2 - u_3 + \hbar)$$

$$\mathrm{Stab}(p_3)|_{p_3} = (u_1 - u_3)(u_2 - u_3)$$

Product of repelling bundle

p_1 outside support of p_2

$$\begin{pmatrix} (u_1-u_2+\hbar)(u_1-u_3+\hbar) & & \\ 0 & (u_1-u_2)(u_2-u_3+\hbar) & \\ 0 & 0 & (u_1-u_3)(u_2-u_3) \end{pmatrix}$$

Smallness: rest are linear in u_i

$$\begin{pmatrix} (u_1-u_2+\hbar)(u_1-u_3+\hbar) & & \\ 0 & (u_1-u_2)(u_2-u_3+\hbar) & \\ 0 & 0 & (u_1-u_3)(u_2-u_3) \end{pmatrix}$$

Next: (4)

$$\left(\mathrm{Stab}(p_1)\Big|_{p_1}\right)_{u_1=u_2} = \left(\mathrm{Stab}(p_1)\Big|_{p_2}\right)_{u_1=u_2}$$

$$(u_1-u_2+\hbar)(u_1-u_3+\hbar)\Big|_{u_1=u_2} = \hbar(u_1-u_3+\hbar)\Big|_{u_1=u_2}$$

2 options:

$$\left(\mathrm{Stab}(p_1)\Big|_{p_2}\right)_{u_1=u_2} = \begin{cases} \hbar(u_1-u_3+\hbar \\ \hbar(u_2-u_3+\hbar) \end{cases}$$

2 options:

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$$(u_1-u_2+\hbar)(u_1-u_3+\hbar)\big|_{u_1=u_3} = \begin{cases} \hbar(u_1-u_2+\hbar \\ \hbar(u_3-u_2+\hbar \end{cases}$$

$$\hbar\,(u_{1,2}-u_3+\hbar)\big|_{u_2=u_3} = \hbar\,(u_{1,3}-u_2+\hbar)\big|_{u_2=u_3}$$

$$\Rightarrow \quad u_{1,2} = u_{1,3}$$

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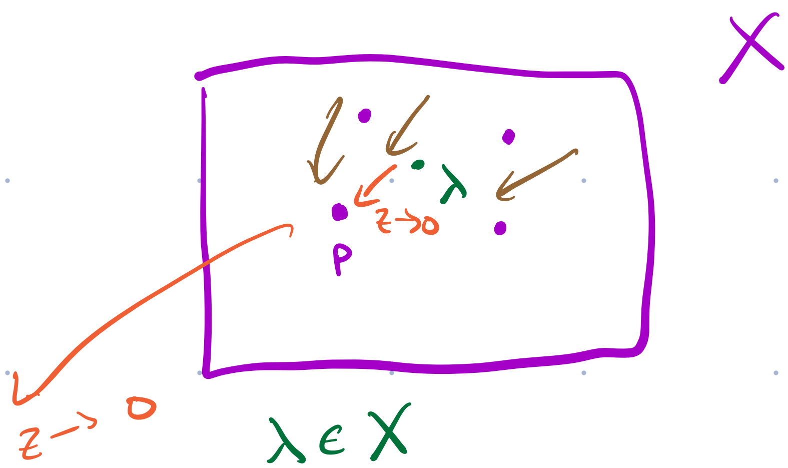

$\lambda \in X$

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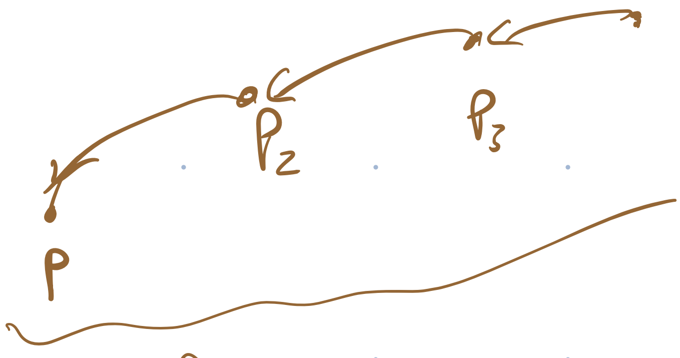
$\mathbb{C}[T]$

$\cup$

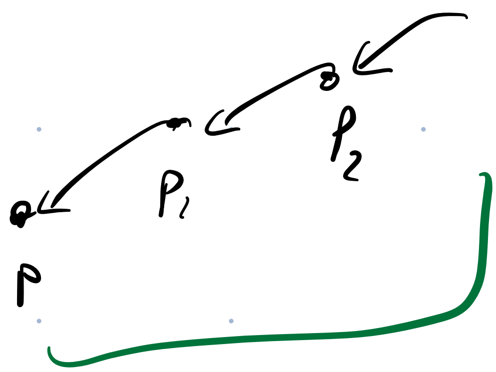
1

$Attr_{\sigma}^{f}(p)$ minimal set invariant wrt closure of $Attr(p)$



full attracting set

First axiom says (1) $Stab_{\sigma}(p)$ is supported at $Attr_{\sigma}^{f}(p)$



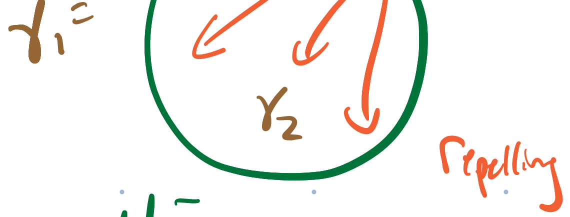
Only nontrivial restrictions on full attracting set

$$(2) \quad \underbrace{Stab_{\sigma}(p)\big|_{p}}_{\text{Equiv Coh Class}} = \pm e(N_p^-) \in H_T(p) = \mathbb{C}[T]$$

e: Euler Class

N_p^-: "repelling" bundle

Euler class



N_p^- – vector space w/ torus action scaling

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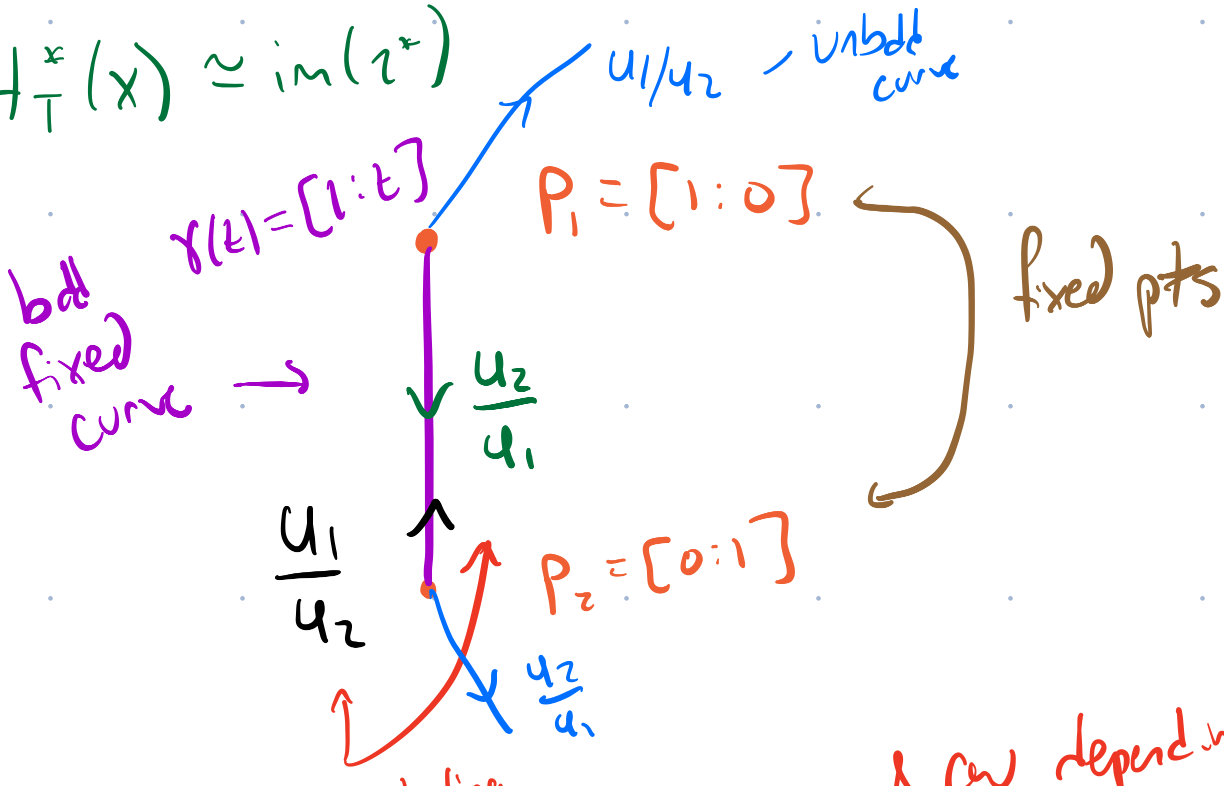

rotating spheres look ccw & cw depending on what you're looking

Really $(p_1, 0) \in T^*\mathbb{P}^1$

$$\mathrm{im}(\iota^*) = \{ (f_1, f_2) \in \mathbb{Q}[u_1, u_2]^2 \mid u_1 - u_2 \mid f_1 - f_2 \}$$

Choose cocharacter

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Ex)

$$\lim_{z \to 0} \sigma(z) \cdot [1:1] = \lim_{z \to 0} (z, z^2) \cdot [1:1]$$

$$= \lim_{z \to 0} [1 : z] = [z : z^2]$$

Copy of $\mathbb{C}$ affine coordinates

$$= [1 : 0] = p_1$$

$$\gamma \subseteq \mathrm{Attr}_\sigma(p_1)$$

$$p_2 \in \overline{\gamma} \subseteq \mathrm{Attr}_\sigma^f(p_1)$$

$$p_1 > p_2$$

Axioms of Stable Envelope

$$\mathrm{Stab}_\sigma : H_T^*(X^T) \longrightarrow H_T^*(X) \quad \begin{array}{l} \xrightarrow{\ \cdot|_{p_1}\ } \mathbb{Q}[u_1,u_2] \\ \xrightarrow{\ \cdot|_{p_2}\ } \mathbb{Q}[u_1,u_2] \end{array}$$

$$H_T^*(X^T) \cong \mathbb{Q}[u_1,u_2] \oplus \mathbb{Q}[u_1,u_2]$$

$$[p_1] = (1,0) \qquad (0,1) = [p_2]$$

$$\left(\mathrm{Stab}_\sigma(p)|_{p'} = 0 \text{ where } p' > p \right)$$

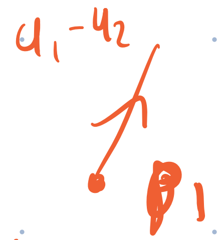
1) $\mathrm{Supp}(\mathrm{Stab}_\sigma(p)) \subseteq \mathrm{Attr}^f_\sigma(p)$

2) $\mathrm{Stab}_\sigma(p)|_p = \pm e(N_-) = \pm \prod_{\text{weights in } N_-} w$

3) $\deg(\mathrm{Stab}_\sigma(p)|_{p'}) < \frac{1}{2}\dim X$

$p' < p$

e.g.



$$e(N_-) = u_1 - u_2$$

$$\mathrm{Stab}_\sigma(p_1)\big|_{p_1} = u_1 - u_2$$

$$\mathrm{Stab}_\sigma(p_2)\big|_{p_2} = u_1 - u_2$$

$$\mathrm{Stab}_\sigma(p_2)\big|_{p_1} = 0$$

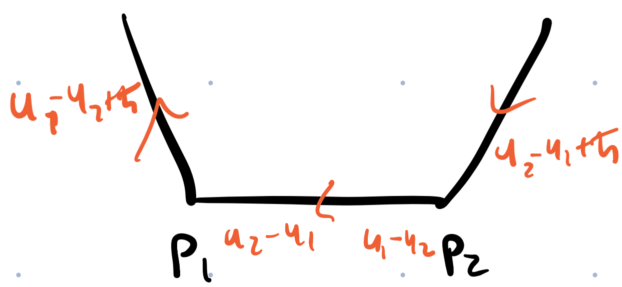
$$\deg\left(\mathrm{Stab}_\sigma(p_1)\big|_{p_2}\right) < \tfrac{1}{2}\dim X = 1$$

$$\mathrm{Stab}_\sigma(p_1)\big|_{p_2} = n$$

$$u_1 - u_2 \,\Big|\, \mathrm{Stab}(p_1)\big|_{p_1} - \mathrm{Stab}(p_1)\big|_{p_2}$$

$$u_1 - u_2 \,\big|\, (u_1 - u_2) - n$$

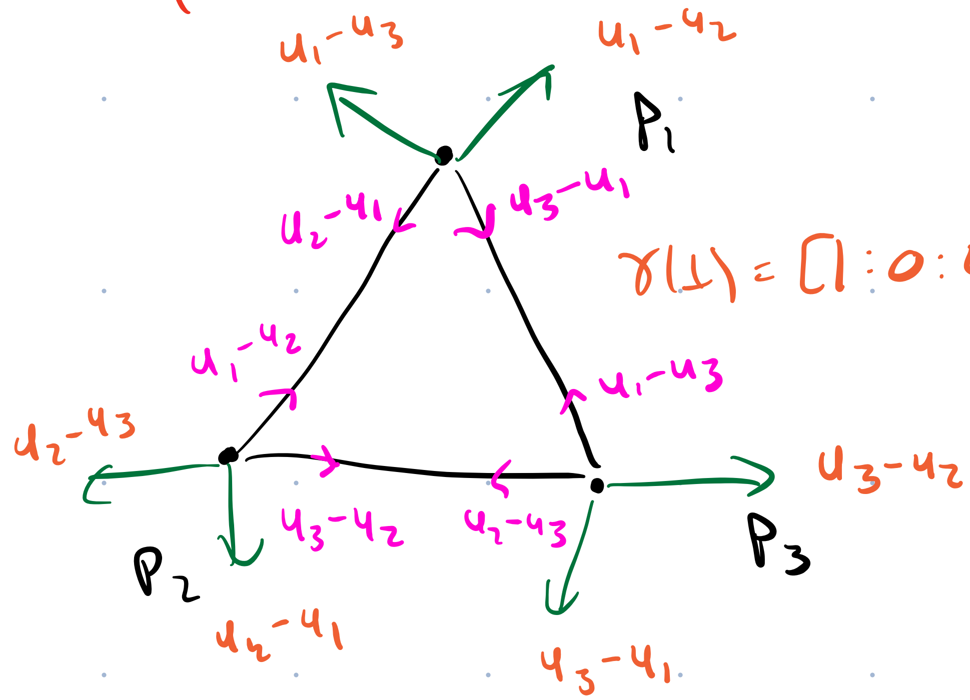
$$\Rightarrow n = 0$$



$$\Rightarrow \begin{pmatrix} u_1 - u_2 + \hbar & \hbar \\ 0 & u_1 - u_2 \end{pmatrix}$$

Ex $X = T^* \mathbb{P}^2$

$T = (\mathbb{C}^\times)^3$

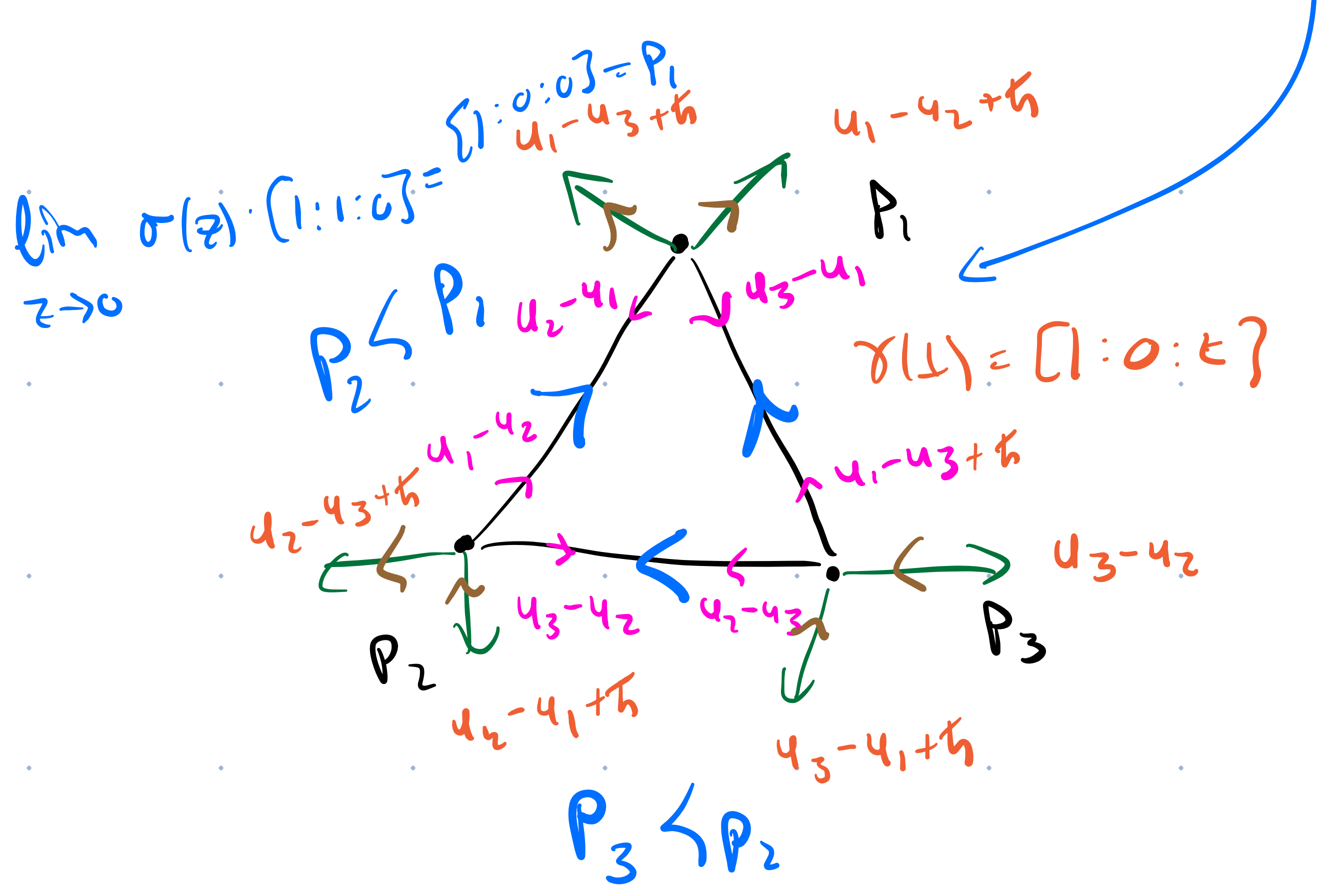


$\gamma(t) = [1:0:t]$

$\sigma : \mathbb{C}^\times \to T \qquad \sigma(z) = (z, z^2, z^3)$

$$\lim_{z \to 0} \sigma(z) \cdot [1:0:1] = P_1$$

$$\Rightarrow \quad P_3 \prec P_1$$



$$\left[\mathrm{Stab}_\sigma(p_i)\big|_{p_j}\right]_{i,j=1}^{3} = \begin{pmatrix} (u_1-u_3+\hbar)(u_1-u_2+\hbar) & \hbar(u_1-u_3+\hbar) & \hbar(u_1-u_2+\hbar) \\ 0 & (u_1-u_2)(u_2-u_3+\hbar) & \hbar(u_1-u_2) \\ 0 & 0 & (u_1-u_3)(u_2-u_3) \end{pmatrix}$$

Row i = $\mathrm{Stab}_\sigma(p_i)$

$$u_2 - u_3 \mid \mathrm{Stab}(p_2)|_{p_2} - \mathrm{Stab}(p_2)|_{p_3}$$

$$\Rightarrow \quad u_2 - u_3 \mid \mathrm{Stab}(p_2)|_{p_3}$$

Gluing condition

$$F|_{p_1} = F|_{p_2}$$

Extra action of $\mathbb{C}^\times_\hbar$, scales symplectic form

Scale fibe

$$\deg_A \mathrm{Stab}(p_i)|_{p_j} < \frac{1}{2} \dim X$$

$$T = A \times \mathbb{C}^\times_\hbar$$

scales symplectic form

R - matrix

$$R = (\mathrm{Stab}_-)^{-1}\,\mathrm{Stab}_+$$

reverse arrows

lower Δ

upper Δ

Open Problem :

$$\int_X \mathrm{Stab}(p) = \sum_{q \in X^T} \frac{\mathrm{Stab}(p)|_q}{e(T_q X)} \Bigg|_{u_1 = \cdots = u_n = 0}$$

$$\in \mathbb{Z}$$

T^*Flag

What is compute for T^* Partial flag

Ex

(first row of matrix)

$$\int_{T^*\mathbb{P}^1} \mathrm{Stab}(p_1) = \frac{u_1-u_2+\hbar}{(u_1-u_2+\hbar)(u_2-u_1)} + \frac{\hbar}{(u_1-u_2)(u_2-u_1+\hbar)}$$

$$= \left.\frac{-u_2+u_1-\hbar+\hbar}{(u_1-u_2)(u_2-u_1+\hbar)}\right|_{u_1=u_2=0}$$

$$= \frac{1}{\hbar}$$

$$= \hbar^{-1}\cdot \underbrace{1}_{\text{integer}}$$

$T^*\mathbb{P}^n$

$$\begin{array}{ccc} & 1 \quad 1 & \\ 1 & 2 & 1 \end{array}$$

T^*Gr unknown

$$H^*(T^*\mathbb{P}^n) \simeq H^*(\mathbb{P}^n) = \frac{\mathbb{Q}[c, u_1, \ldots, u_n, \hbar]}{\prod (c-u_i)}$$

Cohomology class

that restricts
to every pt, so easy guess
$c|_{p_i} = u_i$

"Weight function"

$$w_i(c, u_1, \ldots, u_n, t_n)$$

$$w_i(c, \ldots, t_n)|_{p_i} = w_i(c = u_j\, u_1 \ldots u_n\, t_n)$$

$$= \mathrm{Stab}(p_i)|_{p_j}$$

$c|_{p_i} = u_i$

$T^*\mathbb{P}^1$ $\quad w_2 = -(c - u_1)$

$$w_1 = c - u_2 + t_n$$

Questions to ask:

- Where do "unbounded curves" come from?

- Why is there the given directions in Unbounded curve : to preserve ω

$\omega = \sum dp_i \wedge dq_i = \sum d(\frac{1}{x})p_i \wedge d(xp_i)$

- In $T^*\mathbb{P}^1$ what is $\mathrm{Attr}_{\mathbb{C}}(0) / \mathrm{Attr}_{\mathbb{C}}(\infty)$?

- How do I think about $\mathbb{C}[\mathrm{Lie}\, T]$?

- What does "degree" mean in the context of cohomology?

- What does the polarization do?

9/18 Reese

$$\mathbb{P}^2 \ni [x:y:z] \quad \Bigg\} \quad [\lambda_1 x : \lambda_2 y : \lambda_3 y]$$
$$\circlearrowleft$$
$$(\mathbb{C}^\times)^3 \quad (\lambda_1, \lambda_2, \lambda_3)$$

$\exists$ induced action on $T^*\mathbb{P}^2$, in local coord there is formula

$$\underset{\overset{\shortparallel}{A}}{(\mathbb{C}^\times)^3} \circlearrowright T^*\mathbb{P}^2 \quad \text{preserving symplectic form}$$

$$T = (\mathbb{C}^\times)^3 \times \mathbb{C}^\times_\hbar \circlearrowright T^*\mathbb{P}^2$$

Locally, T^*X has coordinates

$$(\underbrace{q_1, \dots, q_n}_{\text{base space}}, \underbrace{p_1, \dots, p_n}_{\text{differentials of coord}}) \quad \text{where} \quad p_i = dq_i$$

different d

In this nbhd $\omega = \sum dp_i \wedge dq_i$

$$\text{"}A \cdot \omega = \sum d(\ \ ) \wedge d(\lambda_i q_i)$$
$$= \sum d\tfrac{1}{\lambda_i}(q_i) \wedge \lambda_i (dq_i)$$

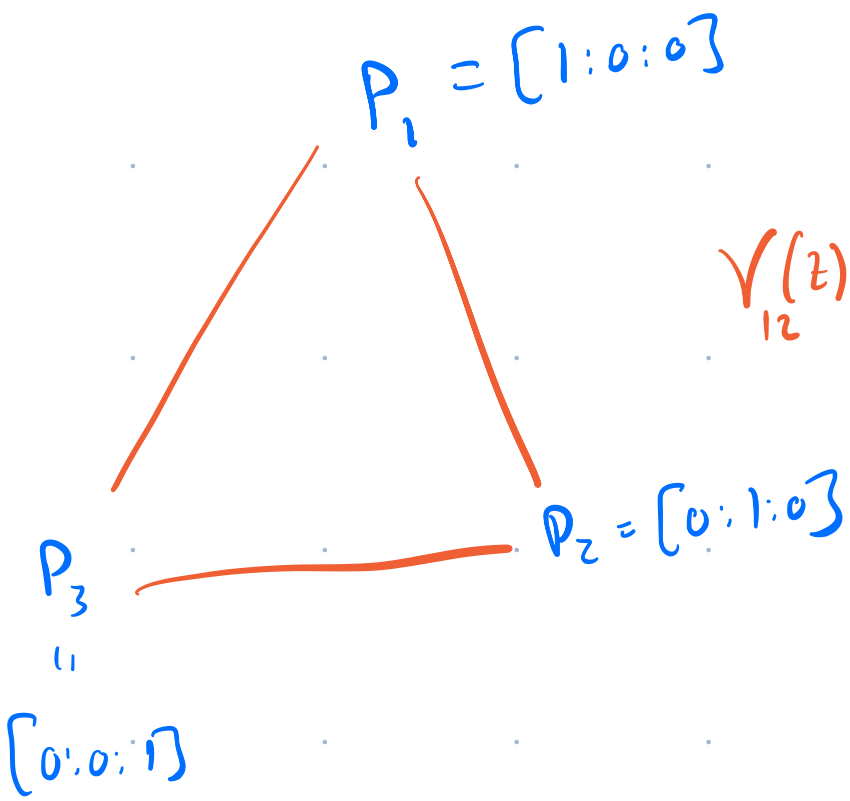
$= \omega^{n}$

$$H_T^\bullet(T^*\mathbb{P}^2) = \left\{ (f_1, f_2, f_3) \in \mathbb{C}[u_1, u_2, u_3, \hbar]^3 \;\middle|\; \begin{array}{l} u_1 - u_2 \mid f_1 - f_2 \\ u_2 - u_3 \mid f_2 - f_3 \\ u_1 - u_3 \mid f_1 - f_3 \end{array} \right\}$$

Moment Graph:

$\exists$ 3 Fixed pts of T-action

— fixed curve



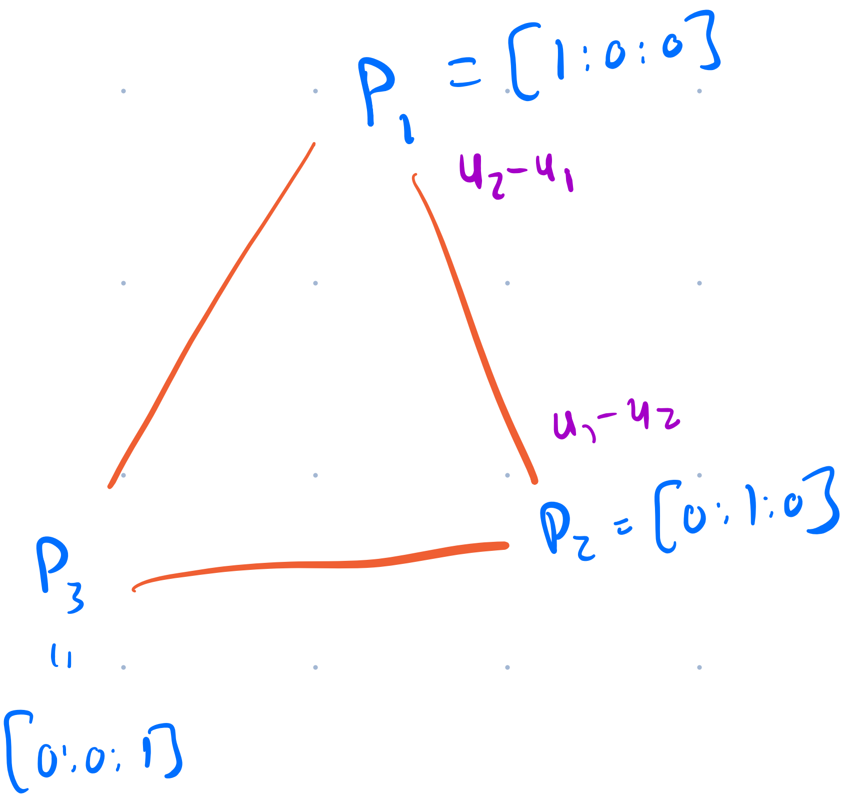
$\gamma_{12}(t) = [1:t:0]$

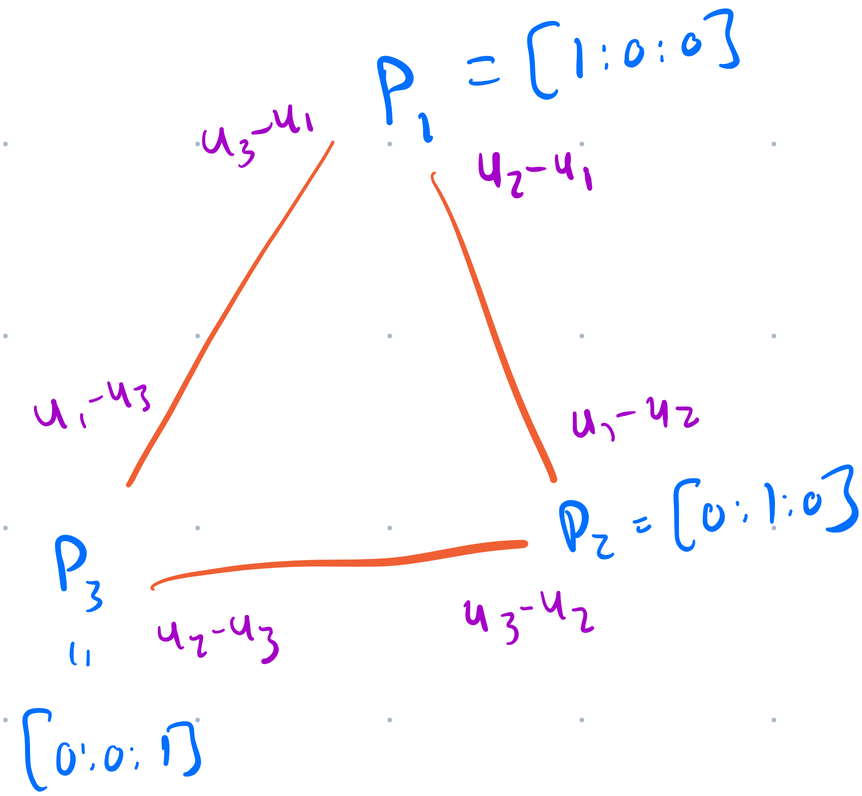
Near P_1 in the direction of P_2, coord are

$$[z_1 : z_2 : 0]$$

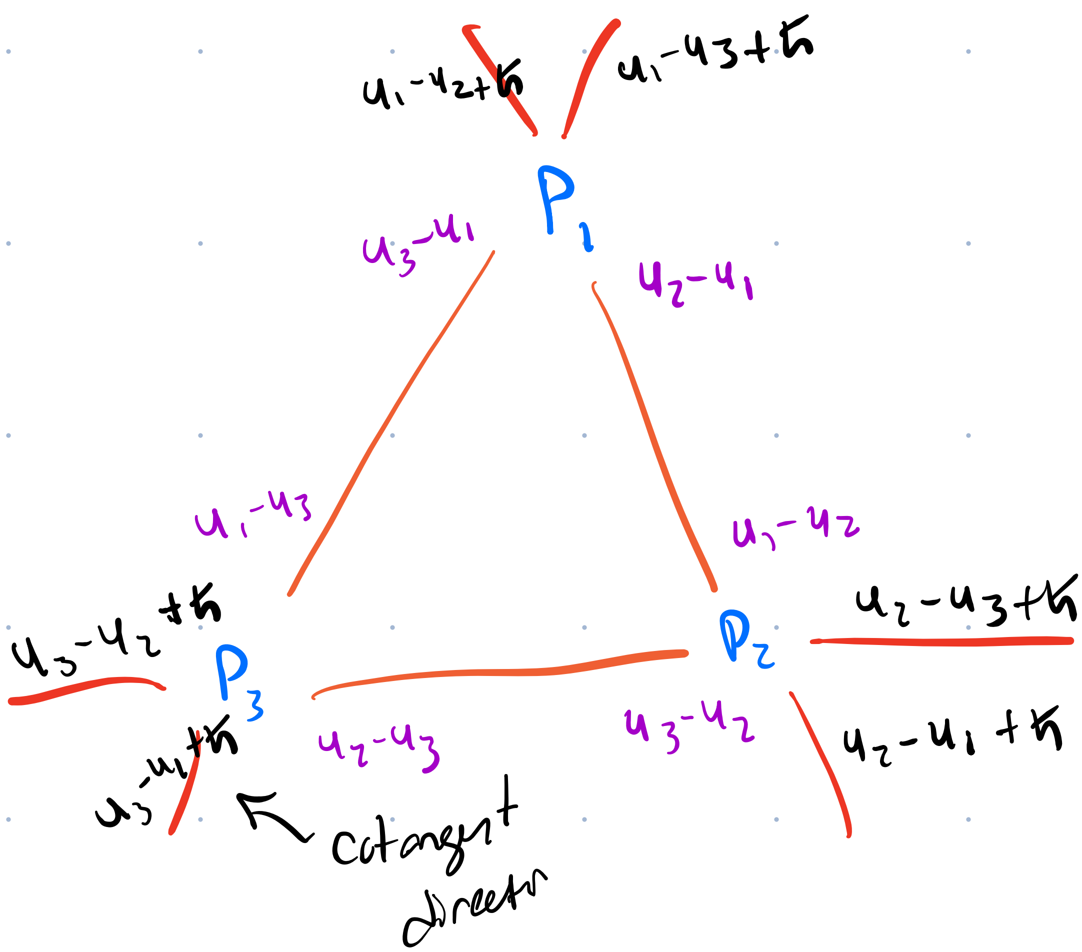
$$= \left[1 : \frac{z_2}{z_1} : 0\right]$$

$$T \cdot \left[1 : \frac{z_2}{z_1} : 0\right] = \left[1 : \frac{\lambda_2}{\lambda_1} \frac{z_2}{z_1} : 0\right]$$





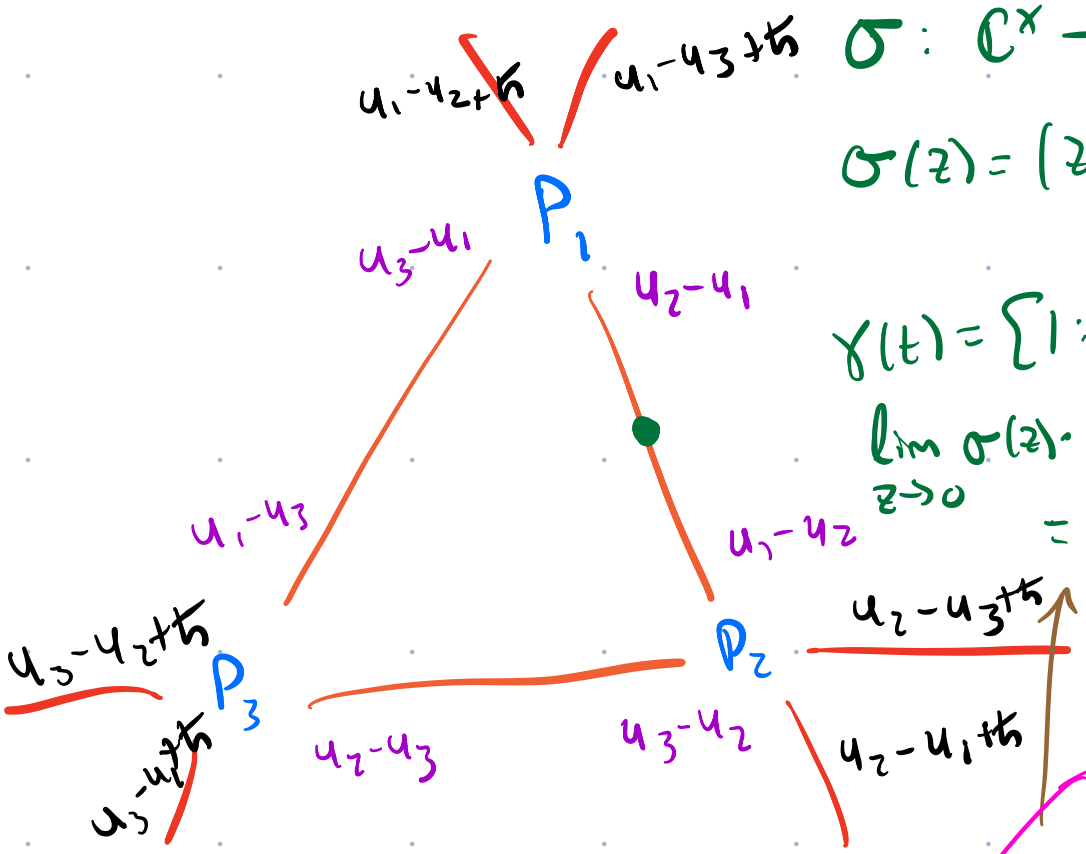
CoTangent space is 4 dim near each pt



Cocharacter

$\sigma : \mathbb{C}^\times \to T$

$\sigma(z) = (z, z^2, z^3, 1)$



$\gamma(t) = [1 : t : 0]$

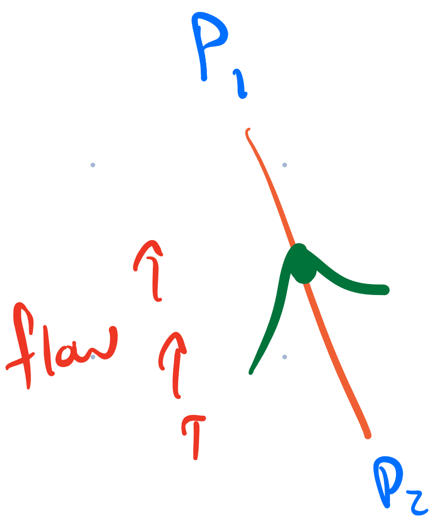
$$\lim_{z\to 0} \sigma(z)\cdot[1:t:0] = \lim_{z\to 0}[z : z^2 t : 0] = \lim_{z\to 0}[1 : zt : 0] = [1:0:0] = P_1$$

Not affine space no limits!

affine chart

so limits works

So,

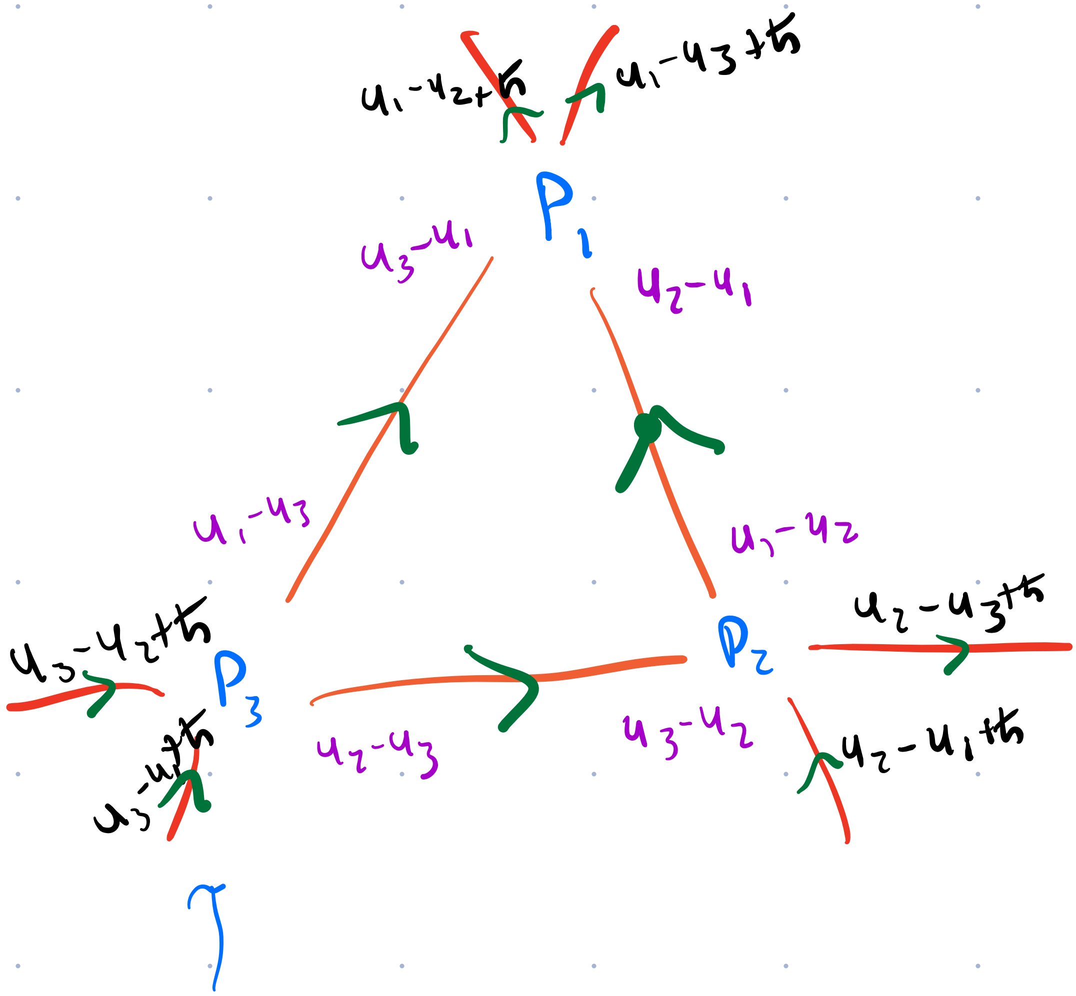


is the attracting direction

Since on the curve $\gamma(t)$

$$\lim \sigma(z)\cdot\gamma(t) = P_1$$

$z \to 0$

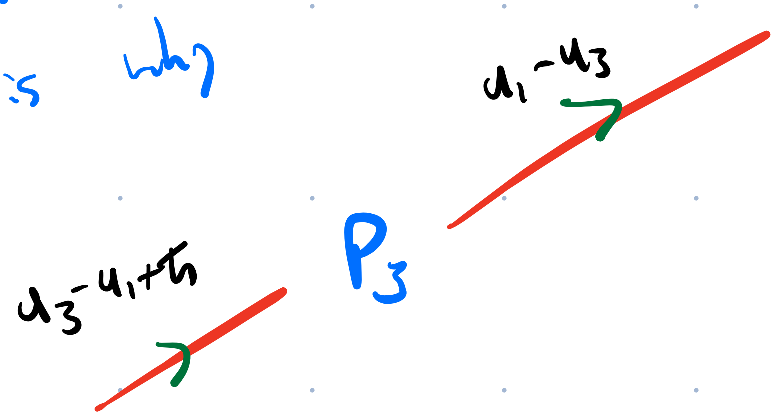


w/o $\hbar$

these curves are the same as A-rep (same weight)

so flow in same direction

this is why



Stable Envelopes:

X has fin many fixed pts / curves

$$\mathrm{Stab}: H_T^\bullet(X^T) \longrightarrow H_T^\bullet(X)$$

map of $H_T^\bullet(pt)$- modules

$$\Rightarrow \quad H_T^\bullet(X^\wedge) \simeq \bigoplus H_T^\bullet(pt)$$
$$= \bigoplus \mathbb{C}[u_1, \dots, u_d, \hbar]$$

P_i have classes $(0, \dots, 0, 1, 0, \dots, 0)$

$\mathrm{Stab}(p_i) \in H_T^\bullet(X)$

Also maps

$$H_T^\bullet(X) \xrightarrow{|_{p_i}} H_T^\bullet(pt)$$

$$\mathrm{Stab}(p_i)\big|_{p_j}$$

Axioms

1) $\mathrm{Supp}(\mathrm{Stab}(p_i)) \subset \mathrm{Attr}_\sigma^F(p_i)$

2) $\mathrm{Stab}(p_i)|_{p_i} = e(N_-)|_{p_i}$

repelling directions

3) $\mathrm{Stab}(p_i)|_{p_j} < \frac{1}{2} \dim X$

$p_j \subset \mathrm{Attr}_\sigma^F(p_i)$

4) $\mathrm{Stab}(p_1) = (f_1, f_2, f_3)$

$u_1 - u_2 \mid f_1 - f_2$

$\iff f_1|_{u_1=u_2} = f_2|_{u_1=u_2}$

$$\mathrm{Stab}(p_1)|_{p_1} = (u_1-u_2+\hbar)(u_1-u_3+\hbar)$$

$$\mathrm{Stab}(p_2)|_{p_2} = (u_1-u_2)(u_2-u_3+\hbar)$$

$$\mathrm{Stab}(p_3)|_{p_3} = (u_1-u_3)(u_2-u_3)$$

Product of repelling bundle

p_1 outside support of p_2

$$\begin{pmatrix} (u_1-u_2+\hbar)(u_1-u_3+\hbar) & & \\ 0 & (u_1-u_2)(u_2-u_3+\hbar) & \\ 0 & 0 & (u_1-u_3)(u_2-u_3) \end{pmatrix}$$

Smallness: rest are linear in u_i

$$\begin{pmatrix} (u_1-u_2+\hbar)(u_1-u_3+\hbar) & & \\ 0 & (u_1-u_2)(u_2-u_3+\hbar) & \\ 0 & 0 & (u_1-u_3)(u_2-u_3) \end{pmatrix}$$

Next: (4)

$$\left(\mathrm{Stab}(p_1)\Big|_{p_1}\right)_{u_1=u_2} = \left(\mathrm{Stab}(p_1)\Big|_{p_2}\right)_{u_1=u_2}$$

$$(u_1-u_2+\hbar)(u_1-u_3+\hbar)\Big|_{u_1=u_2} = \hbar\,(u_1-u_3+\hbar)\Big|_{u_1=u_2}$$

2 options:

$$\left(\mathrm{Stab}(p_1)\Big|_{p_2}\right)_{u_1=u_2} = \begin{cases} \hbar\,(u_1-u_3+\hbar \\ \hbar\,(u_2-u_3+\hbar) \end{cases}$$

2 options:

$$\left(\mathrm{Stab}(p_1)\Big|_{p_1}\right)_{u_1=u_3} = \left(\mathrm{Stab}(p_1)\Big|_{p_3}\right)_{u_1=u_3}$$

$$(u_1-u_2+\hbar)(u_1-u_3+\hbar)\Big|_{u_1=u_3} = \begin{cases} \hbar(u_1-u_2+\hbar \\ \hbar(u_3-u_2+\hbar \end{cases}$$

$$\hbar\,(u_{1,2}-u_3+\hbar)\Big|_{u_2=u_3} = \hbar\,(u_{1,3}-u_2+\hbar)\Big|_{u_2=u_3}$$

$$\Rightarrow \quad u_{1,2}=u_{1,3}$$

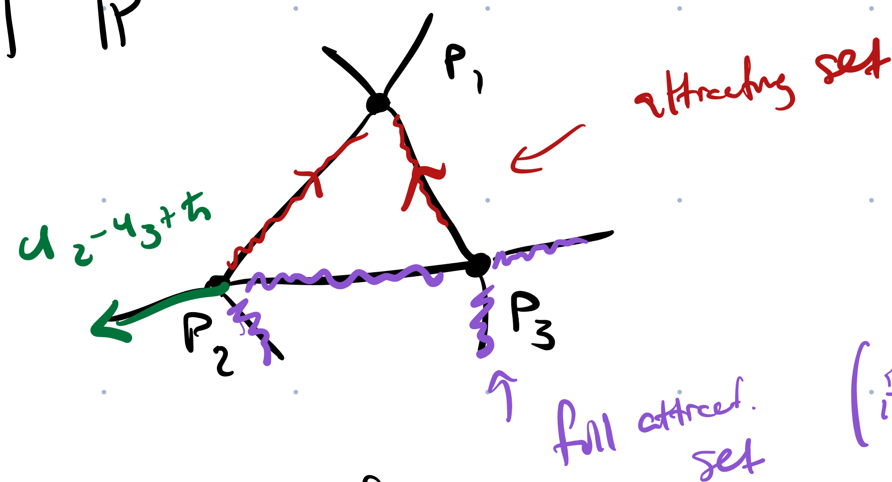
10/2 Alex

$$\mathrm{supp}(\mathrm{Stab}_\sigma(p)) \subseteq \mathrm{Attr}^f_\sigma(p)$$

$$H^*_T(X) \xrightarrow{P} H^*_T(X \setminus \mathrm{Attr}^f_\sigma(p))$$

$$P(\mathrm{Stab}_\sigma(p)) = 0$$

Ex] $T^*\mathbb{P}^2$



(its closure of the red part)

$$\mathrm{Supp}(\mathrm{Stab}(p_1)) \subset \underline{\mathrm{Attr}^f_\sigma(p_1)}$$

$$\mathrm{Stab}(p_1)\big|_{p_2}$$

Claim: By support axiom, $u_2 - u_3 + \hbar \mid \mathrm{Stab}(p_1)\big|_{p_2}$

Gysin Sequence:

$$\begin{array}{c} E \\ \downarrow \pi \\ B \end{array} \qquad E \text{ - rank } k \text{ bundle} \qquad E_0 = E \setminus \pi^{-1}(0)$$

$$\begin{tikzcd} H^i(E, E_0) \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & H^i(E) \arrow[r, "p"] \arrow[d, phantom, "\cong" rotate=90] & H^i(E_0) \arrow[r] & H^{i+1}(E, E_0) \arrow[r] & \cdots \\ H^{i-k}(B) \arrow[r, "\cdot e(E)"'] & H^i(B) & & & \end{tikzcd}$$

Thom iso

as E retracts onto B

$$\operatorname{im}(\cdot e(E)) = \operatorname{Ker}(p) = \{ f \in H^i(E) \mid \operatorname{supp} f \subseteq B \}$$

Things supported on B are multiples of the Euler class $e(E)$

So Pick B to be 3-dim mfd containing the 3 directions at P_2 pointing into the attracting set

$$\underbrace{\operatorname{Attr}^f(p_1)}_{\text{not mfd its not smooth}}$$

Pick E to be a line bundle whose fibre at p_2 is the direction not feeling $\mathrm{Attr}^f(p_1)$

$$\Big((u_1 - u_2 + \hbar)(u_1 - u_3 + \hbar) \qquad \overbrace{(u_2 - u_3 + \hbar) n \hbar}^{\text{degree 2 if we include } \hbar}$$

when $u_1 = u_2$

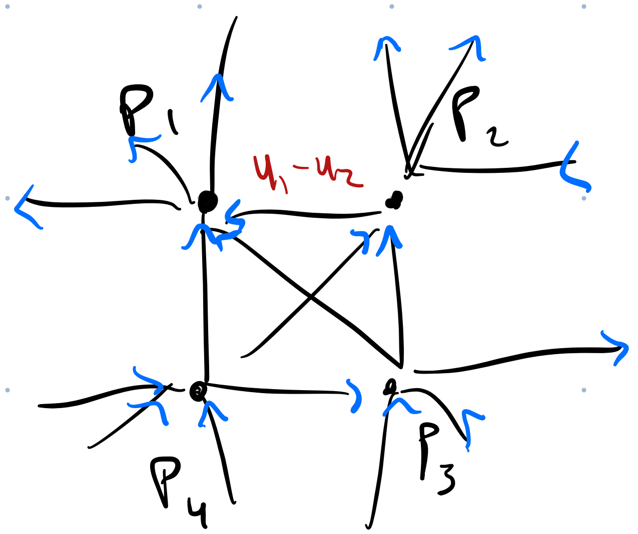
$$(u_2 - u_3 + \hbar) n \hbar = \hbar (u_1 - u_3 + \hbar)$$

$$\Rightarrow n = 1$$

Also homogeneous

Reese

$T^*\mathbb{P}^3$



$\exists$ arrow $p_i \to p_j$ when $i > j$

the weight on that arrow $u_j - u_i$

the cotangent weight $u_i - u_j + \hbar$

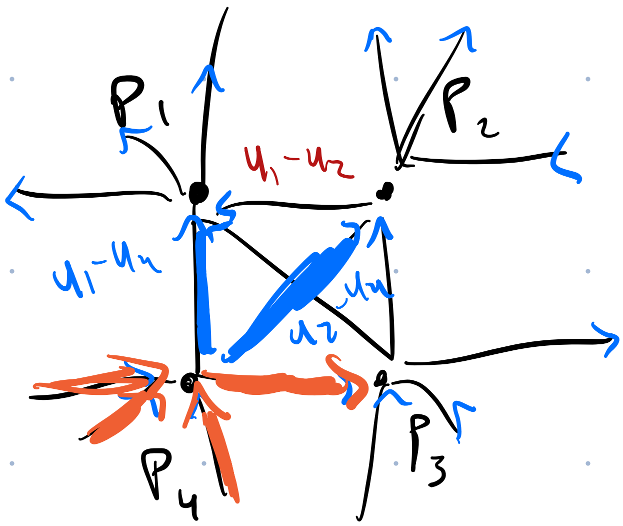
$$\mathrm{Stab}(p_1)\big|_{p_1} = (u_1-u_2+\hbar)(u_1-u_3+\hbar)(u_1-u_4+\hbar)$$

$$\mathrm{Stab}(p_2)\big|_{p_2} = (u_1-u_2)(u_2-u_3+\hbar)(u_2-u_4+\hbar)$$

$$\mathrm{Stab}(p_3)\big|_{p_3} = (u_1-u_2)(u_1-u_3)(u_3-u_4+\hbar)$$

$$\mathrm{Stab}(p_4)\big|_{p_4} = (u_1-u_2)(u_1-u_3)(u_1-u_4)$$

$$\mathrm{Stab}(p_3)\big|_{p_4} \sim (u_1-u_4)(u_2-u_4)\,\hbar$$



$$\underbrace{\left(\mathrm{Stab}(p_3)\big|_{p_4}\right)\Big|_{u_3=u_4}}^{\mathrm{Att}^f_\sigma(p_3)} = \left(\mathrm{Stab}(p_3)\big|_{p_3}\right)\Big|_{u_3=u_4}$$